MASS TRANSFER IN THE FLOW OF A THIN WAVY FILM OF LIQUID

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Waves almost always arise on the surface of a liquid film flowing down an inclined surface as a result of instability. These waves may have an appreciable effect on interphase transfer processes. Thus, with the desorption of hard-to-dissolve gases in the films, the mass-transfer coefficient may be increased by more than 100% due to waves [1, 2].

The study [3], using an approximation of a thin diffusive boundary layer near a free surface, obtain a similarity solution to a diffusion problem for a wavy film which was described parametrically in the form of integrals. It was analyzed in detail, but the investigators showed that the ratio of the integral mass flows in the wavy and nonwavy films approaches a constant value.

Such a solution was also obtained and use was made of the example of sinusoidal waves in [4] to analyze the solution in detail. It was shown that when the phase velocity of the wave is greater than the liquid velocity on the surface of the wavy film — the subcritical flow regime — mass transfer is increased due to the presence of transverse velocity pulsations. Moreover, as the liquid velocity at the wave crests (which is maximal at these points) approaches the phase velocity, the sections of the surface separated by the crests (we will henceforth refer to them as cells) become less closely connected in the diffusive sense: It is more difficult for liquid to move from one cell containing a more dilute gas solution to another cell with a more concentrated solution.

The present study investigates the effect of the wave form on mass transfer in the subcritical flow regime. We also examine the critical case, when the velocity of the surface at the wave crests is equal to the phase velocity of the wave.

The hydrodynamic problem of several nonlinear waves on a film, observed in experiments, has not yet been solved. Thus, as models investigators have taken the profiles of steady traveling waves [5] from the solution of an equation describing the behavior of long-wave perturbations on a film with Re ~ 1. These profiles coincide with the profiles observed experimentally in [6]. The amplitude and velocity of the waves was given. The profile of longitudinal velocity was assumed to be similitudinous:

$$u = 1.5u_{av}(x,\ t)[2y/h(x,\ t) - (y/h(x,\ t))^2]. \tag{1}$$

Here u_{av} is the film velocity averaged over the cross section of the film; y is the normal coordinate, reckoned from the wall to the free surface; x is the longitudinal coordinate; h is the instantaneous thickness of the film. The assumption of similarity is consistent with the experimental findings in [7].

In the approximation of a thin boundary layer adjacent to the free surface, the dimensionless diffusion equation has the form

$$\frac{\partial\theta}{\partial\bar t} + \bar u\frac{\partial\theta}{\partial\bar x} + \bar v\frac{\partial\theta}{\partial\bar y} = \frac{1}{\mathrm{Pe}\,\varepsilon}\frac{\partial^2\theta}{\partial\bar y^2}, \tag{2}$$

where Pe = q/D is the Peclet number; q is the mean flow rate of the liquid; $\theta = (\tilde c - c_h)/(c_0 - c_h)$ is the dimensionless concentration; c, c_0 and c_h are, respectively, the running, initial (at x = 0), and surface (y = h) concentrations; $\bar u$ is the normal component of velocity near the surface; $\varepsilon = \langle h\rangle/\lambda \ll 1$; $\langle h\rangle$ is the mean thickness of the film; λ is the wavelength.

In Eq. (2) and below, we use the dimensionless variables

$$\bar u = u\langle h\rangle/q,\ \bar v = v\lambda/q,\ \bar x = x/\lambda,\ \bar y = y/\langle h\rangle,\ \bar t = tq/\lambda\langle h\rangle$$

and the boundary conditions

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$$\theta(y = h) = 0,\ \theta(x = 0) = \theta(y = -\infty) = 1.$$

We will henceforth omit the sign indicating dimensionlessness. From the continuity equations and (1) for steady traveling waves with $h = h(x - ct)$ we have the following expressions for u and v:

$$u = 1.5(c - (c - 1)/h),\ v = (u - c)\partial h/\partial x - (y - h)\partial u/\partial x. \quad (3)$$

In the new variables

$$x = x,\ \xi = x - ct,\ z = h - y$$

and with the use of Eq. (3), Eq. (2) can be rewritten in the form

$$u\frac{\partial\theta}{\partial x} + (u - c)\frac{\partial\theta}{\partial\xi} - z\frac{du}{d\xi}\frac{\partial\theta}{\partial z} = \frac{1}{\mathrm{Pe}\,\varepsilon}\frac{\partial^2\theta}{\partial z^2}. \quad (4)$$

Equation (4) is solved with the boundary and initial conditions

$$\theta(z = 0) = 0,\ \theta(x = 0) = \theta(z = \infty) = 1. \quad (5)$$

As was shown in [4], the solution of Eq. (4) satisfying conditions (5) is written in the form

$$\theta(z, x, \xi) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\,d\eta,\ \eta = z/\delta(x, \xi), \quad (6)$$

where $\delta(x, \xi)$ is determined from the expression

$$\delta(x, \xi) = 2\left(\int_0^x \frac{(u - c)^2}{u}\,dx/\mathrm{Pe}\,\varepsilon\right)^{1/2} \Big/ (c - u). \quad (7)$$

The integral in (7) is taken along the characteristic

$$x + \int_{\xi_0}^{\xi} u\,d\xi/(c - u) = 0, \quad (8)$$

where ξ_0 is the point of intersection of this characteristic with the axis x = 0. Thus, ξ_0 is the "number" of the characteristic. Consequently, we will henceforth refer to the given characteristic as the ξ_0-characteristic.

Since $u = u(\xi)$, we change over in (7) from the variable x to the variable ξ, considering that they are not independent but instead belong to the ξ_0-characteristic (8):

$$\delta(x, \xi) = 2\left(\int_{\xi}^{\xi_0}(c - u)\,d\xi/\mathrm{Pe}\,\varepsilon\right)^{1/2} \Big/ (c - u). \quad (9)$$

Finally, we have the following expression from (6) and (9) for the instantaneous local dimensionless flow over the wave surface

$$j(x, \xi) \equiv \frac{\partial\theta}{\partial z}\Big|_{z=0} = (c - u)\left[\pi\int_{\xi}^{\xi_0}(c - u)\,d\xi/\mathrm{Pe}\,\varepsilon\right]^{-1/2}. \quad (10)$$

Averaging (10) along the wave gives us the mean local flow at the point X:

$$\bar{j}(x) = \int_0^1 (c - u)\left[\pi\int_{\xi}^{\xi_0}(c - u)\,d\xi/\mathrm{Pe}\,\varepsilon\right]^{-1/2} d\xi. \quad (11)$$

To find the local flow at any point (x, ξ), it is necessary to calculate the integral in Eq. (10). For this, it is sufficient to use (8) to determine the ξ_0-characteristic passing through this point.

In the subcritical case ($c > u_{max}$), the integrand function in (8) is continuous, and the ξ_0-characteristic crosses the entire region $\{0 \le x \le \infty,\ -\infty < \xi < \xi_0\}$, i.e., with movement along the ξ_0-characteristic as $x \to \infty$, $\xi \to -\infty$. The half-line $\{\xi = \xi_1,\ x \ge 0\}$ intersects all of the characteristics with numbers $\xi_0 \ge \xi_1$. In this case, from Eq. (10) we have

$$\lim_{x\to\infty} j(x, \xi_1) = 0.$$

When the velocity of the liquid at the wave crest equals the phase velocity ($c = u_{max}$ $u(n)$), the integrand function at the points $\xi = n$ vanishes, and if ξ_0 lies in the range $(n - 1, n)$, then the corresponding ξ_0-characteristic as a whole lies within the half-band $\{0 \le x \le \infty,\ n - 1 < \xi \le n\}$. With movement along the ξ_0-characteristic as $x \to \infty$, $\xi \to n - 1$. The half-line $\{\xi = \xi_1,\ x \ge 0\}$ intersects only those characteristics for which $\xi_1 \le \xi_0 < n$. Thus, it is apparent that, in contrast to the subcritical case, the upper limit of the integral in Eq.(10) is finite, and the local flow is affected only by the points of the given wave cell. In fact, since for a given point (x_1, ξ_1) there is the relation $n - 1 < \xi_1 \le \xi_0 \le n$, then, considering that $u(\xi) = u(\xi + 1)$, from (10) we obtain

$$j(x, \xi) = (c - u([\xi]))\left[\pi \int_{[\xi]}^{[\xi_0]} (c - u)\, d\xi/\mathrm{Pe}\,\varepsilon\right]^{-1/2}, \tag{12}$$

where [ ] represents the fractional part of the number.

For sufficiently large values of x and finite values of ξ, it follows from (12) that

$$j(x,\xi) \approx (c - u([\xi]))\left[\pi \int_{[\xi]}^{1} (c - u)\, d\xi/\mathrm{Pe}\,\varepsilon\right]^{-1/2}. \tag{13}$$

Thus, throughout almost the entire range $(n - 1, n)$, with large values of x the instantaneous local flow becomes a function only of ξ, and the mean local flux, accordingly, becomes nearly constant. The exception is the neighborhood of the end points of the interval. It can be seen from (8) that with almost any finite value of x, as $[\xi] \to 0$, the value of $[\xi_0]$ also approaches 0. In this case, from (8) we have

$$x = \int_{[\xi]}^{[\xi_0]} u\, d\xi/(c - u) = |u \approx c - u''(0)[\xi]^2/2| = \frac{2c}{u''(0)}\left(\frac{1}{[\xi_0]} - \frac{1}{[\xi]}\right), \tag{14}$$
$$[\xi_0] = 2c[\xi]/(2c + u''(0)\,x[\xi]).$$

From (10) and (14) we obtain

$$j(x, \xi) = [\mathrm{Pe}\,\varepsilon c(1 + b)^3/(\pi x(1 + b + b^2/3))]^{1/2}, \tag{15}$$

where $b = u''(0)x[\xi]/2c$.

As $[\xi] \to 1$, we have an indeterminate form in (13). Evaluating this form, we obtain Eq. (15) for j, in which we need to replace b by $b_1 = u''(0)x(1 - [\xi])/2c$.

From (15) we obtain the physically obvious result $j(x, n) \sim (c/x)^{1/2}$. In fact, it can be seen from (3) that $v = 0$ and $u = c$ at the points $\xi = n$, and diffusive flow at these points over the distance x should be the same as in a plane film flowing downward at the velocity u.

Integrating Eq. (11), we obtain the complete dimensionless mass flow (Sherwood criterion)

$$\mathrm{Sh} = \int_0^x \int_0^1 (c - u)\left[\pi \int_{\xi}^{\xi_0} (c - u)\, d\xi'/\mathrm{Pe}\,\varepsilon\right]^{-1/2} d\xi\, dx = 1 - \bar{\theta}(x), \tag{16}$$

where $\bar{\theta}(x)$ is the flow rate mean concentration in the section x.

The effect of the waves on mass transfer is characterized by the ratio of the mass flows on wavy and nonwavy films [4]:

$$\mathrm{Sh}/\mathrm{Sh}_0 = (3\langle h\rangle x/2h_0)^{-1/2} \int_0^x \int_0^1 (c - u)\left[\pi \int_{\xi}^{\xi_0} (c - u)\, d\xi'/\mathrm{Pe}\,\varepsilon\right]^{-1/2} d\xi\, dx. \tag{17}$$

If assumption (1) is valid, then for equal flow rates we have the following expression for the ratio of $\langle h \rangle$ to the mean thickness of the nonwavy film h_0

$$\langle h\rangle/h_0=\left(\int_0^1 (c(h-1)+1)\,d\xi/h^2\right)^{1/3}.$$

For a purely sinusoidal wave $h = 1 + a\cos 2\pi\xi$, we have constructed the instantaneous local mass flow across the free surface and the mean local flow in Fig. 1 (curves 1 and 2, respectively). The curves were calculated for the values c = 2 and a = 0.4. Curve 3 corresponds to the waveless case. Here and subsequently, we chose the moment of time when the wave crests wre located at points with integral values of the coordinate x. In Figs. 1-3, the values of local flux were plotted in an arbitrary vertical scale.

It can be seen from Fig. 1 that for several wavelengths the instantaneous local flow is nearly a periodic function, while the mean local flow is almost constant. There is then a sudden decrease in the flow, and again for several wavelengths the instantaneous flow is almost period. Such behavior of the curves can be explained as follows [4]. For a purely sinusoidal wave, the local flow has only one maximum per wavelength. It is located in the neighborhood of the trough, so the liquid with a reduced concentration is also located here. When this liquid reaches the trough of the next (left) cell, there is a sudden decrease in the local flow in that trough. The time of transport of any section of liquid along the surface over a distance corresponding to one wavelength is determined by the integral $\int_0^1 d\xi/(c-u)$, so such a decrease first occurs at the distance $x = c\int d\xi/(c-u) - 1$.

When the maximum velocity u_{max} approaches the phase velocity of the wave c, the value of x increases and in the critical case becomes infinite, i.e., the cells become diffusively independent.

In the case of essentially nonsinusoidal waves, its profile over one wavelength has several maxima and minima. Local flow is extreme in the subcritical case in the neighborhood of each of these maxima and minima for sufficiently large x.

Figure 2 shows the relations for instantaneous (curve 1) and mean (curve 2) local flow for a wave having a velocity c = 2.3 and an amplitude a = 0.6. The form of the wave is shown by curve 3. It is apparent that local flow changes markedly almost between any two wavelengths. Some maxima decay more rapidly than others, and the position of the absolute maximum of local flow relative to the wavelength changes from cell to cell. This occurs because now the sharp reduction in the local maxima is connected not only with the appearance of low-concentration liquid from the adjacent cell at the given point, but also with the arrival at this point of liquid from adjacent maxima in the cell containing the given point.

Figure 3 shows values of the instantaneous local (curves 1 and 3) and mean local (curves 2 and 4) flows for the subcritical (a = 0.5, c = 2.2) and critical (a = 0.636, c = 2.2) cases, respectively for the same given wave form (curve 5). The behavior of curves 1 and 2 is qualitatively similar to the relations shown in Fig. 2. In the critical case, the instantaneous local flow after several wavelengths becomes an almost periodic function. Thus, in the present case (curve 3), the maxima after the second and third wavelengths differ by less than 1%. Accordingly, the mean local flow becomes nearly constant after the third wavelength. It is evident that the integral value of Sh for the critical case increases in direct proportion to x.

Figure 4 shows the dependence ofthe flows for a wave which is close to being a sequence of solitary tones: over most of the wavelength the film has a constant thickness, and only near the crest does film thickness change sharply. Here, a = 0.71 and c = 2.326 — the critical case. The numeration of the curves is the same as in Fig. 2.

In the subcritical case, the dependence of the mean local flow on x is nearly a piecewise-constant function (curves 2 in Figs. 1-3), despite the fact that the instantaneous local flow for nonsinusoidal waves is not nearly periodic (compare curves 1 in Figs. 1-3). Thus, the integral mass flow Sh is nearly a piecewise-linear function of x, which is qualitatively similar to the case of purely sinusoidal waves. It is with this very piecewise-linear character of the function Sh(x) that is associated the linear — with respect to $\sqrt{x}$ — increase in the ratio Sh/Sh_0 on the initial section and its subsequently becoming a constant characteristic of the given wave after decaying oscillations (Fig. 5).

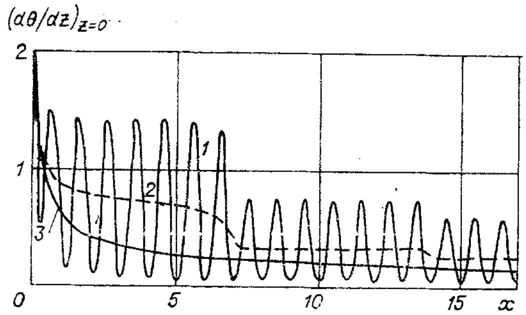

Fig. 1

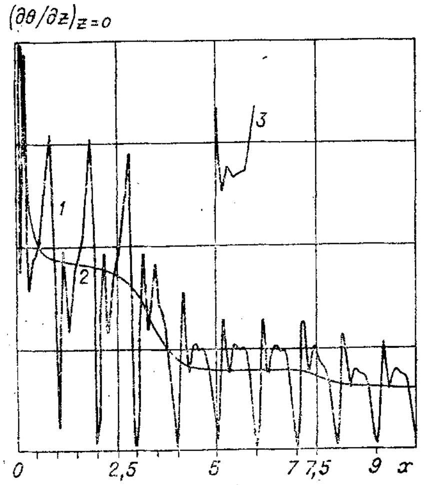

Fig. 2

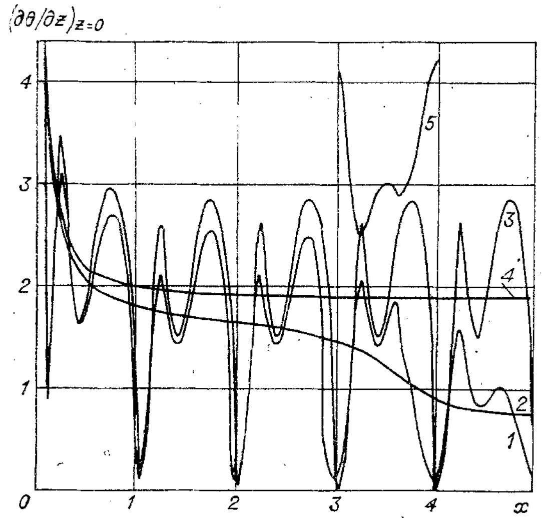

Fig. 3

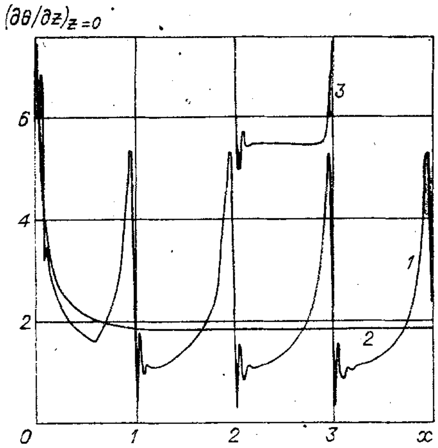

Fig. 4

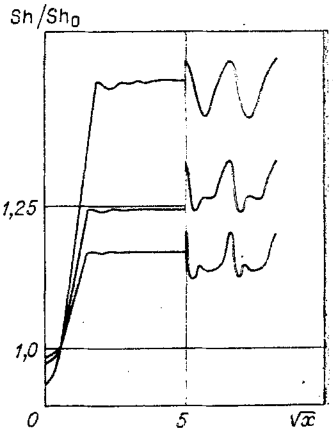

Fig. 5

For a wave of the given form having the velocity c = const, this constant increases as the amplitude a approaches its critical value. In the critical case, Sh(x) is a linear function. Thus, the ratio Sh/Sh_0 also increases linearly with an increase in $\sqrt{x}$ (curve 4 in Fig. 6).

With a fixed value of the wave amplitude, the limiting constant decreases with an increase in phase velocity.

Figure 5 shows the dependence of Sh/Sh_0 for waves having the same velocities (c = 2.2) and amplitudes (a = 0.4). The wave form corresponding to each curves is shown schematically to the right of each curve. With fixed values of amplitude and phase velocity, the asymptotic

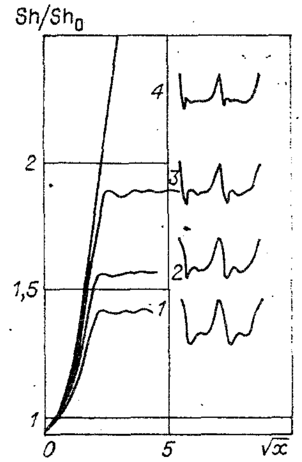

Fig. 6

value of Sh/Sh_0 is less, the more the wave form deviates from sinusoidal. The deterioration in mass transfer for such waves is evidently connected with more rapid decay of the maxima of instantaneous local flow due to their mutual effect on each other. Furthermore, the maxima become narrower, since there is an increase in the percentage of wave sections where $v \approx 0$, and mass transfer takes place only as a result of diffusion.

Thus, the results obtained show that the mass transfer rate is heavily influenced by all three factors: wave velocity, amplitude, and form. Thus, large scatter may be — and is (see, e.g., [1]) — obtained in experimental data without detailed accounting of the wave situation on a film.

These three factors are related to each other in actual flows [6]: The greater the amplitude of the wave, the greater its phase velocity and the more its form will deviate from sinusoidality. Either a decrease of an increase in mass transfer may be seen, depending on which of these three factors exerts the greatest effect.

In particular, the results obtained in [1] now become qualitatively understandable. It was shown in this study that mass transfer decreases with an increase in the period of the wave for fixed Re when waves of different amplitudes are generated. In this instance, wavelength and wave velocity also increase, and the form becomes more nonsinusoidal. Figure 6 shows that within the framework of the given model, such a situation is quite possible. Here, the velocity and amplitude of the wave are, respectively, equal to 2, 2.1, 2.2., 2.326 and 0.4, 0.5, 0.6. 0.71 for curves 1-4. The form of the waves is shown schematically to the right of each curve.

It is evident from the calculations shown in Fig. 6 that if the wavelength of the working section is greater than the maximum transitional section in the given case, we obtain a period dependence for Sh/Sh_0 which is similar to that seen in [1]: Sh/Sh_0 increases with an increase in the period. Such behavior of Sh/Sh_0 cannot be obtained within the framework of other models, such as the complete mixing model.

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STABILITY OF UNSTEADY MOTION OF A VISCOUS FLUID BAND

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A brief derivation is presented in this paper for the small perturbation equations of arbitrary unsteady motion of a viscous incompressible fluid subjected to the action of surface forces. The stability of a viscous fluid band is studied on the basis of the equations obtained.

1. PERTURBATION EQUATIONS

We assume that the functions $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ are the velocity vector and pressure of a certain unsteady motion of a viscous incompressible fluid. The motion is defined in a domain $\Omega_t \subset R^3$ with boundary Γ_t. Within Ω_t, the $\mathbf{u}$, p satisfy the Navier–Stokes equations

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + (1/\rho)\nabla p = \nu\Delta\mathbf{u} + \mathbf{g}(\mathbf{x}, t); \quad (1.1)$$

$$\operatorname{div}\mathbf{u} = 0,\ \mathbf{x} \in \Omega_t,\ t \geqslant 0, \quad (1.2)$$

and on Γ_t the conditions

$$(p_0 - p)\mathbf{n} + 2\rho\nu D(\mathbf{u})\mathbf{n} = 2\sigma H\mathbf{n}; \quad (1.3)$$

$$f_t + \mathbf{u}\cdot\nabla f = 0,\ \mathbf{x} \in \Gamma_t,\ t \geqslant 0, \quad (1.4)$$

where $\nu > 0$, ρ are the constant viscosity and density, $\mathbf{n}$ is the unit vector of the external normal to Γ_t; $\sigma > 0$ is the surface tension coefficient, D is the strain-rate tensor with elements $D_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ (i, j = 1, 2, 3); H is the mean curvature of the surface Γ_t (it is considered that $H > 0$ if Γ_t is convex within the fluid; p_0, $\mathbf{g}$ are the given external pressure and the mass force vector. Condition (1.3) expresses the equality of all forces acting on the free boundary while (1.4) denotes that Γ_t consists of the same particles (the equation $f(\mathbf{x}, t) = 0$ gives the free boundary Γ_t).

At the initial instant

$$\Omega_t|_{t=0} = \Omega,\ \mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}),\ \Gamma_t|_{t=0} = \Gamma \quad (1.5)$$

and the consistency conditions are satisfied

$$\operatorname{div}\mathbf{u}_0 = 0,\ \boldsymbol{\tau}\cdot D(\mathbf{u}_0)\mathbf{n} = 0, \quad (1.6)$$

where τ is an arbitrary vector tangent to Γ.

Let us note that for $\sigma = 0$ the question of single-valued solvability of the problem posed is resolved affirmatively in [1], where $\mathbf{u}$, p, and Γ_t belong to certain Holder classes (see [2] also).

Let the solution $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ of the Navier–Stokes equations satisfying (1.3) and (1.4) on Γ_t, the initial conditions (1.5) and the consistency conditions (1.6) be known in the domain Ω_t. If $(\alpha_1, \alpha_2) \to \mathbf{x}(\alpha_1, \alpha_1, t)|_{t=0}$ is the parametric assignment of the initial surface $\Gamma \in C^3$ while the velocity vector $\mathbf{u}$ is a sufficiently smooth function, then [3] it can be considered that even Γ_t is parametrized by the same parameters (α_1, α_2): $\mathbf{x} = \mathbf{x}(\alpha_1, \alpha_2, t)$.

Let us consider another solution $\tilde{\mathbf{u}}$, $\tilde{p}$ in the domain $\tilde{\Omega}_t$ with the initial field $\tilde{\mathbf{u}}_0 = \mathbf{u}_0 + \mathbf{U}_0$, $\operatorname{div}\mathbf{U}_0 = 0$. Let $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{X}(\mathbf{x}, t)$, $\mathbf{X}$ is the fluid particle displacement vector, $\mathbf{X}|_{t=0} = 0$, such that $\tilde{\Omega}_t|_{t=0} = \Omega$. We set

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